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# Characterizing the result of the division of fuzzy relations

P. Bosc *, O. Pivert, D. Rocacher

*IRISA-ENSSAT, Université de Rennes 1, BP 80518, 22305 Lannion Cedex, France*

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**Abstract**

The role and properties of the division operator are well known in the framework of queries addressed to regular relational databases. However, Boolean queries may turn out to be too restrictive to answer some user needs and it is desirable to consider extended queries by introducing preferences inside selection conditions. In this paper, the extension of the division operator is investigated in the context of graded relations, i.e., whose tuples are weighted. Several interpretations of the division are possible and they mainly depend on the roles of the grades attached to tuples of input relations. Their properties are examined in the perspective of a characterization of the result obtained as a quotient, similarly to that obtained for the division of two integers.


*Keywords:* Databases; Relational model; Relational algebra; Division operation; Flexible queries; Fuzzy relations; Fuzzy implications; Quotient

## 1. Introduction

The database domain is an important field of research and development and many works aim at enriching database management systems (DBMSs) capabilities. The context of the research reported in this paper is the expression of flexible queries, i.e., where preferences intervene in selection conditions instead of Boolean predicates. This view is illustrated by the query: "find the affordable restaurants located close to the seashore". In such

* Corresponding author. Tel.: +33 2 96 46 90 45; fax: +33 2 96 37 01 99.
*E-mail address:* Bosc@enssat.fr (P. Bosc).

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a situation, discrimination among restaurants has to take into account both the price of the menu(s) and the location of the restaurants (and optionally levels of importance attached to each of these two criteria).

Several works devoted to the expression and the interpretation of fuzzy queries in the relational framework [4] have been undertaken (in particular [1,9,11]). Selection, projection, Cartesian product, join as well as set-oriented operations have been studied in order to take into account levels of preference. On the contrary, the division operation has not been so much investigated [2,3,5,7,8,10,12,13]. These different extensions have various motivations and contexts, in particular depending on the nature of the relations involved and the meaning of degrees associated with tuples. In [2,3,5,7,10,13], fuzzy relations (i.e., relations whose tuples are assigned a weight expressing their compliance with a given fuzzy concept) are considered with different definitions of the division. In [8], a special attention is paid to the case where the degrees attached to the tuples are imprecisely known, while in [12], the authors deal with relations where attributes may be imprecise (represented as possibility distributions) and tuples are weighted by degrees of possibility. Last, let us mention that in [8,13], the authors consider the case where the universal quantifier involved in the division operation is weakened into "most", which comes down to make a kind of approximate division.

In the remainder of this paper, the division of fuzzy relations in the sense of relations whose tuples are weighted, is investigated. The principal objective is to discuss the properties of the result delivered by a division operation. Indeed, this result depends on the approach adopted for the extension of the division as it is mentioned in the works reported in [2,3,7]. One would like to determine if the result obtained is a quotient in the sense of the double property which holds when the division of two integers is performed, i.e., the largest integer whose product with the divisor remains lower than (or equal to) the dividend. As it will be recalled later, an analogous property is valid for the division of regular relations which delivers a relation which is the largest one whose Cartesian product with the divisor relation is included in the dividend relation. The key point behind this work is of a semantic nature, because a negative answer in the case of a division of fuzzy relations would mean that the term division is not quite appropriate.

The rest of the paper is organized as follows. In Section 2, the definition of the division of regular relations is recalled as well as the two characteristic properties of a quotient. The principle for adapting the division to fuzzy relations, which relies on the notion of a degree of inclusion (instead of a usual Boolean inclusion) is described in Section 3. The next section is devoted to the study of the division of fuzzy relations in a logical framework, i.e., the degree of inclusion is based on fuzzy implications, while Section 5 concerns a cardinality-based approach for defining the degree of inclusion. In Section 6, the meaning of the divisions delivering a quotient is addressed. Finally, Section 7 concludes the paper in two respects: the major results obtained are recalled and some perspectives for future works are outlined.

## 2. Some reminders about the division

The relational division, i.e., the division of relation r whose schema is $R(A, X)$ by relation s whose schema is $S(B)$ where $A$ and $B$ are compatible sets of attributes (i.e., defined on the same domains of values) is defined as:

$$\mathrm{div}(r, s, A, B) = \{x \mid (x \in \mathrm{dom}(X)) \wedge (s \subseteq K_r(x))\} \tag{1}$$

where $K_r(x) = \{a | <a, x> \in r\}$. In other words, an element $x$ belongs to the result of the division of $r$ by $s$ if and only if it is associated in r with *at least all* the values a appearing in $s$. The justification of the term "division" assigned to this operation relies on the fact that a property similar to that of the quotient of integers holds. Indeed, the resulting relation t obtained with expression (1) has the double characteristic of a quotient:

$$\text{prod}(s, t) \subseteq r \tag{2a}$$

$$\forall t1, (t1 \supset t) \Rightarrow (\text{prod}(s, t1) \not\subset r) \tag{2b}$$

$\text{prod}(s, t)$ being the Cartesian product of the two relations $s$ and $t$.

**Proof**

*Case* 1. Neither the result of the division, nor the divisor relation is empty. Let $x$ be an element of $t$ and let a be an element of $s$. Let us suppose that $<x, a>$ does not belong to $r$, then $x$ would not be associated with all the values of $s$ and it would not be in the result of the division of $r$ by $s$, hence inclusion (2a) holds. Now, let us consider relation $t1 = t \cup \{y\}$ $(y \notin t)$. The Cartesian product of $t1$ and $s$ contains a tuple $<y, b>$ which does not belong to $r$, otherwise y would be associated with any value a of $s$ and it would have been in $t$. It follows that property (2b) holds.

*Case* 2. The result of the division is empty but the divisor is not empty. Property (2a) holds since the Cartesian product of $t$ and $s$ is empty and then included in any relation. No element $x$ is associated with all the elements of $s$ and if $y$ is added to $t$, property (2b) does not hold since the Cartesian product of {y} with s involves elements which are not in $r$.

*Case* 3. The divisor $s$ is empty. The solution returned by (1) is the (possibly infinite) set of the values in the domain of $X$. Properties (2a) and (2b) are both satisfied since the Cartesian product of $t$ and $s$ is empty and $t$ cannot be augmented. □

**Remark.** When the divisor is empty, the theoretical solution of the division is the entire domain of $X$. In practice, such a solution cannot be computed since the domains of the attributes are not represented (and are thus unknown) in database systems. To overcome this problem, a solution is to adapt the definition of the division by constraining the possible elements of the result to belong to the dividend relation. So, the practical computation of the result can be performed even if the divisor is empty and the definition of the division becomes:

$$\text{div}(r, s, A, B) = \{x | (x \in \text{proj}(r, X)) \wedge (s \subseteq K_r(x))\} \tag{3}$$

where $\text{proj}(r, X)$ stands for the projection of relation $r$ over attribute $X$ defined as

$$\text{proj}(r, X) = \{x | \exists t \wedge (t \in r) \wedge (t \cdot X = x)\} \tag{4}$$

The characterization of a quotient is changed into:

$$\forall x, (x \in t) \Rightarrow (\text{prod}(s, \{x\})) \subseteq r) \tag{5a}$$

$$\forall t1, (t1 = t \cup \{x\}) \wedge (x \in \text{proj}(r, X)) \Rightarrow (\text{prod}(s, \{x\}) \not\subset r)). \tag{5b}$$

Expressions (5a) and (5b) express the fact that the relation t resulting from the division is a quotient, i.e., *the largest relation* whose Cartesian product with the divisor returns a result smaller than or equal to the dividend (according to regular set inclusion).

**Example 1.** Let us take a database involving the two relations order ($o$) and product ($p$) with respective schemas O(np, store, qty) and $P$(np, price). Tuples $<n, s, q>$ of $o$ and $<n, pr>$ of $p$ state that the product whose number is $n$ has been ordered to store $s$ in quantity $q$ and that its price is $pr$. The query aiming at retrieving the stores which have been ordered all the products priced under \$127 in a quantity greater than 35, can be expressed thanks to a division as

$$\mathrm{div}(o\text{-}g35, p\text{-}u127, \{\mathrm{np}\}, \{\mathrm{np}\})$$

where relation $o$-$g$35 corresponds to pairs $(n, s)$ such that product $n$ has been ordered to store $s$ in a quantity over 35 and relation $p$-$u$127 gathers products whose price is under \$127. From the following extensions of relations $o$ and $p$:

| *o* | np | Store | qty |
|---|---|---|---|
| | 15 | 32 | 50 |
| | 12 | 32 | 68 |
| | 34 | 32 | 49 |
| | 26 | 32 | 78 |
| | 26 | 7 | 120 |
| | 78 | 7 | 30 |
| | 46 | 7 | 65 |
| | 12 | 7 | 96 |

| *p* | np | Price |
|---|---|---|
| | 15 | 102 |
| | 4 | 200 |
| | 12 | 87 |
| | 26 | 59 |
| | 78 | 345 |
| | 34 | 258 |
| | 46 | 175 |

the relations $o$-$g$35 and $p$-$u$127 obtained are

| *o*-*g*35 | np | Store |
|---|---|---|
| | 15 | 32 |
| | 12 | 32 |
| | 34 | 32 |
| | 26 | 32 |
| | 26 | 7 |
| | 46 | 7 |
| | 12 | 7 |

| $p$-$u$127 | np |
|---|---|
| | 15 |
| | 12 |
| | 26 |

whose division using formula (3) leads to a result made of the single element {32} since store 7 is not associated with product 15. This result satisfies expressions (2a) and (2b), or alternatively (5a) and (5b) since:

$$p\text{-}u127 \times \{32\} = \{<15,32>,<12,32>,<26,32>\}$$

is (here strictly) included in $o$-$g$35 and any element different from {32} would not satisfy the inclusion.

## 3. Approaches to the division of fuzzy relations

### 3.1. Fuzzy queries and fuzzy relations

The context considered now is that of flexible queries where conditions call on preferences instead of Boolean criteria. The answer to such a query is made of a set of elements rank-ordered according to their compliance with the preferences. From now on, predicates of flexible queries are assumed to be modeled by fuzzy sets [1] and fuzzy relations are used instead of regular ones.

Formally, a fuzzy relation is defined as a fuzzy subset of the Cartesian product of domains of values. Hence, a fuzzy relation $r$ whose schema is $R(A, B, C)$ is made of a set of weighted triples denoted by $\mu_r(t)/t$, where $t = <a, b, c>$ and $\mu_r(t)$ stands for the membership degree of $t$ in relation $r$, i.e., its compatibility with the fuzzy concept associated with this relation. It is worth noticing that a regular relation is just a special case of a fuzzy relation where the degree attached to every tuple equals 1.

A flexible query is made of operations applying to fuzzy relations whose result is also a fuzzy relation. Such operations are obtained through a natural extension of the usual algebraic operators (unary and binary) and the most common are presented hereafter. The usual selection of relation r by means of the Boolean condition "cond" defined as:

$$\text{sel}(r, \text{cond}) = \{t | (t \in r) \wedge \text{cond}(t)\}$$

is extended to a fuzzy relation r and a fuzzy condition $f$-cond by

$$\mu_{\text{sel}(r,f-\text{cond})}(t) = \top(\mu_r(t), \mu_{f-\text{cond}}(t))$$

where $\top$ stands for a triangular norm intended for generalizing the conjunction. Examples of triangular norms are the minimum and the product. Each norm $\top$ is associated with a co-norm $\perp$ generalizing the disjunction. Each pair of norm and co-norm is linked by the following law:

$$\perp(a, b) = 1 - \top(1 - a, 1 - b)$$

where $(1 - a)$ stands for the negation of $a$. The co-norm associated with the minimum (respectively product) is the maximum (respectively the sum minus the product). The projection of relation $r$, whose schema is $R(X)$, onto the subset of attributes $Y$ of $X$ defined by formula (4) is extended to a fuzzy relation $r$ by

$$\mu_{\text{proj}(r,Y)}(y) = \sup_{t \in \text{supp}(r) | t.Y=y} \mu_r(t)$$

where supp($r$) denotes the support of the relation $r$, i.e., the regular relation defined as

$$\text{supp}(r) = \{t | \mu_r(t) > 0\}.$$

Set-oriented operations of union, intersection, difference and Cartesian product which are based on conjunctions and/or disjunctions are straightforwardly extended to fuzzy relations. Let us notice that the latter serves as a basis for the join operation and that its input relations may have different schemas. These four operations are defined as follows:

$$\mu_{\text{union}(r,s)}(t) = \bot\, (\mu_r(t), \mu_s(t))$$
$$\mu_{\text{intersection}(r,s)}(t) = \top(\mu_r(t), \mu_s(t)) \tag{6}$$
$$\mu_{\text{difference}(r,s)}(t) = \top(\mu_r(t), 1 - \mu_s(t))$$
$$\mu_{\text{prod}(r,s)}(uv) = \top(\mu_r(u), \mu_s(v)) \tag{7}$$

where $\top$ (respectively $\bot$) denotes a norm (respectively a co-norm).

### 3.2. Principle for extending the division

By analogy with a query calling on a division such as that of Example 1, one may envisage the query aiming at the determination of *the extent* to which any store has been ordered all the *fairly cheap* products in a *high* quantity, which is expressed thanks to a division of fuzzy relations, namely:

$$\text{div}(hq\text{-}o, fcp\text{-}p, \{\text{np}\}, \{\text{np}\})$$

where the degree attached to any tuple of *hq-o* (respectively *fcp-p*) expresses the compatibility of the quantity (respectively price) with high (respectively fairly cheap).

The extension of the division to fuzzy relations is based on the adaptation of formula (3) where:

- the regular inclusion is replaced by a fuzzy one (i.e., a degree of inclusion),
- the expression of the restriction of the calculus to the values present in the dividend accounts for the fact that the divisor is a fuzzy relation.

This yields:

$$\forall x \in \text{proj}(\text{supp}(r, X)), \mu_{\text{div}(r,s,A,B)}(x) = \deg(s \subseteq K_r(x)) \tag{8}$$

where proj(supp($r, X$)) represents the domain of $X$ restricted to those values appearing in the dividend (relation $r$) and $K_r(x)$ is defined as

$$K_r(x) = \{\mu/a | \mu/ < x, a > \in r\}.$$

Several types of degrees of inclusion exist depending on the approach adopted. The logical one is based on:

$$E \subseteq F \iff \forall x \in \mathrm{U}, (x \in E) \Rightarrow (x \in F)$$

where $U$ is the underlying referential. This leads to:

$$\deg(E \subseteq F) = \min_x \mu_E(x) \Rightarrow_f \mu_F(x) \tag{9}$$

where $\Rightarrow_f$ is a fuzzy implication. Another one is founded on cardinalities of (finite) fuzzy sets, namely:

$$E \subseteq F \iff \mathrm{card}(E \cap F) = \mathrm{card}(E)$$

which leads to a degree of inclusion expressing a ratio of cardinalities:

$$\deg(E \subseteq F) = \mathrm{card}(E \cap F)/\mathrm{card}(E) \tag{10}$$

Last, a third approach relies on the set difference operation:

$$E \subseteq F \iff (E - F) = \emptyset$$

which is extended to:

$$\deg(E \subseteq F) = 1 - h(E - F) \tag{11}$$

where $h(G)$ denotes the height of the fuzzy set $G$, i.e., the highest degree of membership attached to its elements.

### 3.3. Characterizing the result of the extended division

Assessing the fact that the result $t$ of the extended division is a quotient entails an adaptation of the double characterization conveyed by formulas (5a) and (5b) in order to take into account that fuzzy relations come into play, which yields:

$$\forall x, (x \in \mathrm{proj}(\mathrm{supp}(r), X) \wedge \mu_t(x) = d) \Rightarrow (\mathrm{prod}(s, \{d/ < x >\})) \subseteq r \tag{12a}$$
$$\forall x, (x \in \mathrm{proj}(\mathrm{supp(r)}, X) \wedge \mu_t(x) = d) \Rightarrow (\forall d1 > d, \mathrm{prod}(s, \{d1/ < x >\}) \not\subseteq r). \tag{12b}$$

It is then necessary to specify the Cartesian product of fuzzy relations as well as the inclusion used in the previous two expressions. As it will be seen, it is of interest to consider a generalized version of the Cartesian product where the conjunction used in formula (7) is not restricted to triangular norms, which leads to:

$$\mu_{\mathrm{prod}(r,s)}(uv) = cnj(\mu_r(u), \mu_s(v)),$$

where *cnj* denotes an extended conjunction operator. As to the inclusion, it is based on the crisp inclusion of fuzzy sets defined by Zadeh, i.e.:

$$E \subseteq F \iff \forall x \in \mathrm{U},\ \mu_E(x) \leqslant \mu_F(x). \tag{13}$$

In the next two sections, the properties of the result of an extended division are studied in terms of satisfaction of properties (12a) and (12b) depending on the definition retained for the division of fuzzy relations, i.e., the underlying inclusion (formulas (9)–(11)).

## 4. A logical view of the division of fuzzy relations

In this section, the extension of the division is studied when fuzzy implications come into play. After some reminders about fuzzy implications in general, the focus is put on the use of $R$-implications and $S$-implications for the division of fuzzy relations. Some connections between these fuzzy implications and two types of conjunction operators (triangular norms and a family of non-commutative conjunctions) are pointed out. This serves as a basis for showing that the division built with the considered fuzzy implications returns a quotient. Then, the relationship existing between the logical and difference-based approaches is addressed. Last, the division of fuzzy relations with various $R$ and $S$ implications is illustrated with a special attention to the fulfillment of properties (12a) and (12b).

### 4.1. Some reminders on fuzzy implications

Fuzzy implications can be classified in two main families satisfying a number of axioms, in particular:

- the decreasing (respectively increasing) monotony with respect to the first (respectively second) argument,
- $(0 \Rightarrow_f a) = 1$,
- $(a \Rightarrow_f 1) = 1$,
- $(1 \Rightarrow_f a) = a$).

The last three axioms guarantee that any such implication generalizes the regular implication, i.e., that a $\Rightarrow_f b$ returns the same result as the regular implication when the antecedent and the conclusion parts take only the values 0 and 1.

An $R$-implication, denoted by $\Rightarrow_{R-i}$, is defined as

$$p \Rightarrow_{R-i} q = \sup\{u \in [0,1] | \top(p,u) \leqslant q\} \tag{14}$$

where $\top$ stands for a continuous triangular norm. Any $R$-implication may be rewritten:

$$p \Rightarrow_{R-i} q = \left| \begin{array}{ll} 1 & \text{if } p \leqslant q \\ f(p,q) & \text{otherwise} \end{array} \right.$$

where $f(p,q)$ expresses a partial satisfaction (a value less than 1) when the threshold $p$ is not reached by the conclusion part $q$. The minimal element of $R$-implications is Gödel implication:

$$p \Rightarrow_{Gö} q = \left| \begin{array}{ll} 1 & \text{if } p \leqslant q \\ q & \text{otherwise} \end{array} \right.$$

which is obtained by choosing $\top(a,b) = \min(a,b)$ in formula (14). It is worth noticing that this implication is purely ordinal since it calls only on the comparison between values. Other representatives of $R$-implications are Goguen implication:

$$p \Rightarrow_{Gg} q = \left| \begin{array}{ll} 1 & \text{if } p \leqslant q \\ q/p & \text{otherwise} \end{array} \right.$$

obtained with $\top(a,b) = a \times b$ in formula (14) and Lukasiewicz implication:

$$p \Rightarrow_{Lu} q = \left| \begin{array}{ll} 1 & \text{if } p \leqslant q \\ 1 - p + q & \text{otherwise} \end{array} \right.$$

obtained with $\top(a,b) = \max(a + b - 1, 0)$.

On the other hand, the material implication defined as

$$p \Rightarrow q = ((\textbf{not } p) \textbf{ or } q)$$

where $p$ and $q$ are Boolean truth values, is also extended by the family of $S$-implications, denoted by $\Rightarrow_{\mathrm{S-i}}$, as follows:

$$p \Rightarrow_{S-i} q = \perp (1-p, q) = 1 - \top(p, 1-q) \tag{15}$$

where $p$ and $q$ take their values in the unit interval $[0,1]$. The most common representatives of this family are Kleene–Dienes implication (minimal element of the family):

$$p \Rightarrow_{K-D} q = \max(1-p, q),$$

Reichenbach implication:

$$p \Rightarrow_{Rb} q = 1 - p + pq,$$

as well as Lukasiewicz implication:

$$p \Rightarrow_{Lu} q = \min(1, 1-p+q)$$

obtained with $\top(a,b) = \min(a,b)$, $\top(a,b) = ab$ and $\top(a,b) = \max(a+b-1, 0)$ respectively in formula (15). Let us notice that Lukasiewicz implication is both an $R$ and an $S$ implication.

### 4.2. Using R and S implications in the division of fuzzy relations

The connection between $R$-implications (denoted by $\Rightarrow_{R-i}$) and conjunctions is due to the very definition of an $R$-implication. Combining expression (8) and the choice of an $R$-implication in (9), the following definition of the division of fuzzy relations is obtained:

$$\forall x \in \mathrm{proj}(\mathrm{supp}(r, X)), \mu_{\mathrm{div}(r,s,A,B)}(x) = d = \inf_s(\mu_s(a) \Rightarrow_{R-i} \mu_r(a,x)). \tag{16}$$

Due to the definition of an $R$-implication (formula (14)), the result delivered by expression (16) is clearly a quotient. Indeed, let us denote by $b$ one of the values of $s$ for which $d$ is obtained, i.e.:

$$d = (\mu_s(b) \Rightarrow_{R-i} \mu_r(b,x))$$

One has:

$$\forall a \in \mathrm{supp}(s), \top(\mu_s(a), d) \leqslant \mu_r(a,x),$$

$$\forall d1 > d, \top(\mu_s(b), d1) > \mu_r(b,x)$$

where $\top$ is the norm used to generate the considered $R$-implication. Since this is true for any $x$ of the dividend, this ensures that the result of the division delivered by formula (16) satisfies properties (12a) and (12b).

On the other hand, it is clear that formulas (15) and (14) differ significantly. Consequently, one cannot expect that formulas (12a) and (12b) hold with a Cartesian product based on the triangular norm used in expression (15) as it is illustrated in the following example.

**Example 2.** Let us consider the following two fuzzy relations $r$ and $s$ whose respective schemas are $R(A, X)$ and $S(B)$:

| r | A | X | μ |
|---|---|---|---|
| | $a1$ | $x$ | 1 |
| | $a2$ | $x$ | 0.8 |

| s | B | μ |
|---|---|---|
| | $a1$ | 1 |
| | $a2$ | 0.5 |

The result $t$ of the division of $r$ and $s$ using Kleene–Dienes implication is

$$\mu_t(x) = \min(1 \Rightarrow_{K-D} 1, 0.5 \Rightarrow_{K-D} 0.8) = \min(1, 0.8) = 0.8.$$

The Cartesian product of t and the divisor s using the largest norm (the minimum) yields:

$$\{0.8/ < a1, x >, 0.5/ < a2, x >\}$$

which is included in the dividend ($r$), but $t$ is not maximal. In effect, the Cartesian product of $t' = 1/ <x>$ with $s$ using the minimum, namely $\{1/<a1, x>, 0.5/<a2, x>\}$ is also included in $r$. The same occurs if Reichenbach implication is used instead of Kleene–Dienes. In that case, the result of the division is $t = \{0.9/<\text{x}>\}$. The Cartesian product of t with $s$ using the minimum:

$$\{0.9/ < a1, x >, 0.5/ < a2, x >\}$$

is included in $r$, but once again not maximal.

These two situations illustrate the fact that, whatever the norm used, the result of the division using those two $S$-implications may not be maximal.

Nevertheless, it has been shown in [6] that any $S$-implication generated by a continuous norm (in the sense of formula (15)) can be rewritten as

$$p \Rightarrow_{S-i} q = \sup\{u \in [0, 1] | \text{ncc}(p, u) \leqslant q\} \tag{17}$$

where $\text{ncc}(a, b)$ is a non-commutative conjunction defined as

$$\text{ncc}(a, b) = 1 - (a \Rightarrow_{R-i} (1 - b)) \tag{18}$$

where the underlying $R$-implication is the one generated by the norm associated with the $S$-implication (according to formula (15)). More precisely, any such non-commutative conjunction has the following properties:

- it coincides with the usual conjunction when $a$ and $b$ are the usual truth values (represented by 0 and 1),
- it is non-commutative,
- 1 is its left-hand side neutral element,
- it is monotonically increasing with respect to both arguments.

**Example 3.** Kleene–Dienes implication can be written:

$$p \Rightarrow_{K-D} q = 1 - \min(p, 1-q) = \sup\{u \in [0,1] | \mathrm{ncc1}(p,u) \leqslant q\} \text{ where } \mathrm{ncc1}(a,b)$$
$$= 1 - (a \Rightarrow_{G\ddot{o}} (1-b)) = \begin{vmatrix} 0 & \text{if } (a+b) \leqslant 1, \\ b & \text{otherwise} \end{vmatrix}$$

Similarly, for Reichenbach implication, one has:

$$p \Rightarrow_{Rb} q = 1 - p(1-q) = \sup\{u \in [0,1] | \mathrm{ncc2}(p,u) \leqslant q\} \text{ with } \mathrm{ncc2}(a,b)$$
$$= 1 - (a \Rightarrow_{Gg} (1-b)) = \begin{vmatrix} 0 & \text{if} (a+b) \leqslant 1, \\ (a+b-1)/a & \text{otherwise.} \end{vmatrix}$$

For Lukasiewicz implication, the non-commutative conjunction operator turns out to be a norm, since, as mentioned previously, this particular implication is also an $R$-implication. More precisely, one has:

$$p \Rightarrow_{Lu} q = \min(1, 1-p+q) = \sup\{u \in [0,1] | \mathrm{ncc3}(p,u) \leqslant q\}$$
$$\text{with } \mathrm{ncc3}(a,b)$$
$$= 1 - (a \Rightarrow_{Lu} (1-b)) = \max(0, a+b-1) \text{ which is indeed a triangular norm.}$$

Last, let us mention that no such operator exists for the maximal $S$-implication generated by the minimal norm:

$$\top_m(a,b) = \begin{vmatrix} a & \text{if } b = 1 \\ b & \text{if } a = 1 \\ 0 & \text{otherwise} \end{vmatrix}$$

which is not continuous.

It is then obvious that if one uses the appropriate non-commutative conjunction operator (in the sense of expression (18)) for the Cartesian product, formulas (12a) and (12b) hold, which means that the result delivered by the division based on $S$-implications is a quotient. It is worth noticing that the order of the arguments of the Cartesian product matters (the divisor is the first operand and the result of the division the second one).

**Remark.** It should be noticed that the semantics of inclusion conveyed by $S$-implications (in general) is not compatible with the intuitive view according to which $\deg(E \subseteq F)$ equals 1 when $E$ is included in $F$ in Zadeh's sense (formula (13)). For example, with Kleene–Dienes and Reichenbach implications, 1 is obtained if and only if the support of $E$ (elements having a strictly positive degree of membership in $E$) is included in the core of $F$ (elements whose membership degree in $F$ is 1), which is much more demanding than the conventional set inclusion based on formula (13).

### *4.3. Difference-based degree of inclusion and S-implications*

Let us come back to the use of the degree of inclusion based on formula (11) for performing the division of two fuzzy relations as specified in formula (8). One has:

$$\forall x \in \mathrm{proj}(\mathrm{supp}(r, X)), \mu_{\mathrm{div}(r,s,A,B)}(x) = \deg(s \subseteq K_r(x)) = 1 - h(s - K_r(x))$$

If one uses the definition of the difference given in formula (6), one gets:

$$\forall x \in \text{proj}(\text{supp}(r, X)), \mu_{\text{div}(r,s,A,B)}(x) = 1 - \max_{a \in \text{supp}(s)} \top(\mu_s(a), 1 - \mu_{K_r(x)}(a))$$
$$= \min_{a \in \text{supp}(s)} 1 - \top(\mu_s(a), 1 - \mu_{K_r(x)}(a)).$$

On the other hand, from formulas (8), (9) and (15), the generic definition of the division based on an $S$-implication is:

$$\forall x \in \text{proj}(\text{supp}(r, X)), \mu_{\text{div}(r,s,A,B)}(x) = \deg(s \subseteq K_r(x)) = \min_{a \in \text{supp}(s)} \mu_s(a) \Rightarrow_{S-i} \mu_{K_r(x)}(a)$$
$$= \min_{a \in \text{supp}(s)} 1 - \top(\mu_s(a), 1 - \mu_{K_r(x)}(a)).$$

This proves that the logical approach based on $S$-implications captures also the semantics of a degree of inclusion built from the standard difference between fuzzy sets according to formulas (6) and (11).

**Example 4.** Let us take the two fuzzy sets:

$$E = \{1/a1, 0.6/a2, 0.3/a3\}, F = \{0.7/a1, 0.4/a2, 1/a3\}.$$

The degree of inclusion of $E$ in $F$ using Kleene–Dienes implication in formula (9) is:

$$\deg(E \subseteq F) = \min(1 \Rightarrow_{K-D} 0.7, 0.6 \Rightarrow_{K-D} 0.4, 0.3 \Rightarrow_{K-D} 1) = \min(0.7, 0.4, 1) = 0.4.$$

On the other hand, the difference of $E$ and $F$ with the norm minimum in formula (6) is:

$$E - F = \{0.3/a1, 0.6/a2\}$$

and

$$h(E - F) = 0.6$$

which equals 1 minus the preceding degree of inclusion (0.4).

The same happens with Reichenbach implication for which the degree of inclusion is:

$$\deg(E \subseteq F) = \min(1 \Rightarrow_{Rb} 0.7, 0.6 \Rightarrow_{Rb} 0.4, 0.3 \Rightarrow_{Rb} 1) = \min(0.7, 0.64, 1) = 0.64$$

and the difference based on the corresponding norm (the product) is:

$$E - F = \{0.3/a1, 0.36/a2\}$$

and

$$h(E - F) = 0.36 = 1 - 0.64.$$

Finally, it appears that the division of fuzzy relations based on any $R$-implication (respectively $S$-implication) generated by a continuous norm, delivers a quotient provided that the Cartesian product involved in the characterization makes use of the norm which serves for generating the $R$-implication through formula (14) (respectively the non-commutative conjunction associated with this $S$-implication via formula (17)).

### 4.4. Illustrations

Let us consider the following two fuzzy relations $r$ and $s$ whose respective schemas are $R(A, X)$ and $S(B)$:

| $r$ | $A$ | $X$ | $\mu$ |
|---|---|---|---|
| | $a1$ | $x$ | 0.7 |
| | $a2$ | $x$ | 0.4 |
| | $a3$ | $x$ | 1 |
| | $a1$ | $y$ | 1 |
| | $a2$ | $y$ | 0.6 |
| | $a3$ | $y$ | 0.2 |

| $s$ | $B$ | $\mu$ |
|---|---|---|
| | $a1$ | 1 |
| | $a2$ | 0.5 |
| | $a3$ | 0.3 |

The result $t$ of the division of $r$ by $s$ is successively computed with different $R$ and $S$-implications. With Gödel implication, it yields:

$$\mu_t(x) = \min(1 \Rightarrow_{Gö} 0.7, 0.5 \Rightarrow_{Gö} 0.4, 0.3 \Rightarrow_{Gö} 1) = 0.4$$
$$\mu_t(y) = \min(1 \Rightarrow_{Gö} 1, 0.5 \Rightarrow_{Gö} 0.6, 0.3 \Rightarrow_{Gö} 0.2) = 0.2.$$

When performing the Cartesian product of $s$ and $t$ with the norm "minimum", one gets the relation:

$$\{0.4/ < a1, x >, 0.4/ < a2, x >, 0.3/ < a3, x >, 0.2/ < a1, y >, 0.2/ < a2, y >, 0.2/ < a3, y >\}$$

which is strictly included in $r$ (formula (12a) holds). It is easy to check that formula (12b) holds as well, because of the presence of the tuples $<a2, x>$ and $<a3, y>$ whose grades equal those they have in $r$.

If Goguen implication is used, the result of the division is:

$$\mu_t(x) = \min_{Gg} 0.7, 0.5 \Rightarrow_{Gg} 0.4, 0.3 \Rightarrow_{Gg} 1) = 0.7$$
$$\mu_t(y) = \min(1 \Rightarrow_{Gg} 1, 0.5 \Rightarrow_{Gg} 0.6, 0.3 \Rightarrow_{Gg} 0.2) = 2/3 \approx 0.67.$$

If the Cartesian product of s and t is performed with the norm "product", one gets the relation:

$$\{0.7/ < a1, x >, 0.35/ < a2, x >, 0.21/ < a3, x >, 0.67/ < a1, y >, 0.33/ < a2, y >, 0.2/ < a3, y >\}$$

which is strictly included in $r$. Here again, it is easy to check that the result of the division is maximal. Then, formulas (12a) and (12b) hold.

Similarly, with Kleene–Dienes implication, the result $t$ of the division of $r$ by $s$ is:

$$\mu_t(x) = \min \Rightarrow_{K-D} 0.7, 0.5 \Rightarrow_{K-D} 0.4, 0.3 \Rightarrow_{K-D} 1) = 0.5$$
$$\mu_t(y) = \min(1 \Rightarrow_{K-D} 1, 0.5 \Rightarrow_{K-D} 0.6, 0.3 \Rightarrow_{K-D} 0.2) = 0.6.$$

When performing the Cartesian product of $s$ and $t$ with the appropriate non-commutative conjunction (ncc1), one gets:

$$\{0.5/ < a1, x >, 0.6/ < a1, y >, 0.6/ < a2, y >\}$$

which is strictly included in $r$. Relation $t$ is maximal because if 0.5 (respectively 0.6) is increased to $0.5^+$(respectively $0.6^+$), the value ncc1(0.5, $0.5^+$) (respectively ncc1(0.5,$0.6^+$)) leads to assign the degree $0.5^+$ (respectively $0.6^+$) to $<a2, x>$ (respectively $<a2, y>$) which is over 0.4 (respectively 0.6) and formula (12b) holds as well.

If Reichenbach implication is used, the result of the division of $r$ by $s$ is:

$$\mu_t(\mathrm{x}) = \min \Rightarrow_{Rb} 0.7, 0.5 \Rightarrow_{Rb} 0.4, 0.3 \Rightarrow_{Rb} 1) = 0.7$$
$$\mu_t(y) = \min(1 \Rightarrow_{Rb} 1, 0.5 \Rightarrow_{Rb} 0.6, 0.3 \Rightarrow_{Rb} 0.2) = 0.76.$$

The Cartesian product of $s$ and $t$ with ncc2 returns the relation:

$$\{0.7/ < a1, x >, 0.4/ < a2, x >, 0.76/ < a1, y >, 0.52/ < a2, y >, 0.2/ < a3, y >\}$$

which is strictly included in $r$. In addition, it is easy to check that relation $t$ is maximal.

Last, let us use Lukasiewicz implication. The result of the division is:

$$\mu_t(\mathrm{x}) = \min(1 \Rightarrow_{Lu} 0.7, 0.5 \Rightarrow_{Lu} 0.4, 0.3 \Rightarrow_{Lu} 1) = 0.7$$
$$\mu_t(\mathrm{y}) = \min(1 \Rightarrow_{Lu} 1, 0.5 \Rightarrow_{Lu} 0.6, 0.3 \Rightarrow_{Lu} 0.2) = 0.9.$$

When performing the Cartesian product of $s$ and $t$ with the associated norm $\top(a, b) = \max(a + b - 1, 0)$, one gets:

$$\{0.7/ < a1, x >, 0.2/ < a2, x >, 0.9/ < a1, y >, 0.4/ < a2, y >, 0.2/ < a3, y >\}$$

which is strictly included in $r$. In addition, $t$ is maximal and once again, formulas (12a) and (12b) hold.

## 5. A cardinality-based approach to the division of fuzzy relations

If formula (10) is used as a basis for extending the division, the definition hereafter is obtained:

$$\mu_{\mathrm{div}(r,s,A,B)}(x) = \mathrm{card}(s \cap K_r(x))/\mathrm{card}(s) = \sum_{a \in \mathrm{supp}(s)} \top(\mu_s(a), \mu_r(a, x)) / \sum_{a \in \mathrm{supp}(s)} \mu_s(a). \quad (19)$$

The question is once again to assess whether or not this view of the division of fuzzy relations delivers a quotient. We will see that this is not the case in general, whatever the type of conjunction used for the Cartesian product, i.e., a norm or a non-commutative conjunction.

Let us consider the extensions of the dividend and divisor relations $r$ and $s$ given hereafter:

| $r$ | $A$ | $X$ | $\mu$ |
|---|---|---|---|
| | $a1$ | $x$ | 0.4 |
| | $a2$ | $x$ | 1 |
| | $a3$ | $x$ | 1 |

| $s$ | $B$ | $\mu$ |
|---|---|---|
| | $a1$ | 1 |
| | $a2$ | 1 |
| | $a3$ | 1 |

Using the minimal norm $\top_m$ (introduced in Example 3) in expression (19), the degree assigned to $x$ by the division is:

$$(\top_m(0.4, 1) + \top_m(1, 1) + \top_m(1, 1))/(1 + 1 + 1) = 0.8.$$

If the Cartesian product of $s$ and $t = \{0.8/x\}$ is performed with the same norm, one gets:

$$\{0.8/ < a1, x >, 0.8/ < a2, x >, 0.8/ < a3, x >\}$$

which is not included in the dividend $r$.

Similarly, let us take the following extensions of the dividend and divisor relations $r$ and $s$:

| $r$ | $A$ | $X$ | $\mu$ |
|---|---|---|---|
| | $a1$ | $x$ | 0.6 |
| | $a2$ | $x$ | 0.1 |
| | $a3$ | $x$ | 0.1 |

| $s$ | $B$ | $\mu$ |
|---|---|---|
| | $a1$ | 1 |
| | $a2$ | 0.3 |
| | $a3$ | 0.3 |

Using the norm "minimum" (which is the largest norm) in expression (19), the degree assigned to $x$ by the division is:

$$(\min(0.6, 1) + \min(0.1, 0.3))/(1 + 0.3 + 0.3) = 0.5.$$

If the Cartesian product of $s$ and $t = \{0.5/x\}$ is performed with the largest non-commutative conjunction ncc1 (the one associated with Kleene–Dienes implication), one gets:

$$\{0.5/ < a1, x >\}$$

which is included in the dividend $r$, but $t = \{0.5/x\}$ is not maximal, since the product of $s$ and $t' = \{0.6/x\}$ using ncc1 would give:

$$\{0.6/ < a1, x >\},$$

which is also included in $r$.

Finally, it appears that using triangular norms or non-commutative conjunctions to perform the Cartesian product:

- the smallest Cartesian product of the divisor and the smallest result of a division may lead to a relation which is not included in the dividend,
- the largest result of a division may not be maximal.

These two facts allow to conclude that this type of division does not comply with properties (12a) and (12b) which characterize a quotient.

*Special case*. When the divisor $s$ is included (in the usual sense, i.e., according to formula (13)) in $K_r(x)$, the fuzzy set of values attached to a given $x$, the degree assigned to $x$ by formula (19) is 1 if the conjunction operator chosen is the minimum. In such a case, whatever the norm used for the Cartesian product, formulas (12a) and (12b) hold. In effect, since 1 is a neutral element for any norm, one has:

$$\mu_{\text{prod}(s,t)}(a, x) = \mu_s(a) \leqslant \mu_r(a, x)$$

and $t$ is obviously maximal since $\mu_t(x) = 1$. This is clearly no longer true either if a non-commutative conjunction is used for the Cartesian product, or if a norm different from the minimum is taken for the division.

**Example 5.** Let us consider the following extensions of the dividend and divisor relations $r$ and $s$:

| $r$ | $A$ | $X$ | $\mu$ |
|---|---|---|---|
| | $a1$ | $x$ | 1 |
| | $a2$ | $x$ | 0.5 |
| | $a3$ | $x$ | 0.8 |

| $s$ | $B$ | $\mu$ |
|---|---|---|
| | $a1$ | 1 |
| | $a2$ | 0.1 |
| | $a3$ | 0.5 |

Using the norm “minimum” in expression (19), the degree assigned to $x$ by the division is:

$$(\min(1,1)+\min(0.1,0.5)+\min(0.5,0.8))/(1+0.1+0.5)=1$$

The Cartesian product of $s$ and $t=\{1/x\}$ with any norm leads to:

$$\{1/<a1,x>, 0.1/<a2,x>, 0.5/<a3,x>\}.$$

This relation is included in the dividend $r$ and the result delivered by the division is obviously maximal. If a non-commutative conjunction is used for the Cartesian product, for instance ncc1 or ncc2, one would get:

$$\{1/<a1,x>, 1/<a2,x>, 1/<a3,x>\}$$

which is not included in the dividend relation $r$.

On the contrary, if the norm chosen in expression (19) is the product (respectively $\top(a,b)=\max(a+b-1,0)$), the degree assigned to $x$ is:

$$(1*1+0.1*0.5+0.5*0.8)/(1+0.1+0.5)=1.45/1.6\approx 0.91$$

$$\text{(respectively } (\max(1+1-1,0)+\max(0.1+0.5-1,0)$$
$$+\max(0.5+0.8-1,0))/(1+0.1+0.5)=1.3/1.6\approx 0.81)$$

The Cartesian product with the largest norm leads to $\{0.91/<a1,x>, 0.1/<a2,x>, 0.5/<a3,x>\}$ (respectively $\{0.81/<a1,x>, 0.1/<a2,x>, 0.5/<a3,x>\}$) which is included in the dividend, but the result of the division is not maximal since, as pointed out before, the Cartesian product of $s$ and $\{1/<x>\}$ is also included in the dividend.

## 6. Semantic aspects of the division of fuzzy relations

The division of fuzzy relations constitutes an enrichment of database query languages which, in addition, turns out to be sound from a theoretical point of view provided that *R*-implications or *S*-implications are used. To illustrate the impact of the choice of a given type of fuzzy implications, we will take into consideration an application context where two relations come into play:

- the dividend describes candidates and their level in various skills,
- the divisor is a profile made of skills as well.

The idea is that this system is addressed queries of the form:

“find the candidates with a high level in all the skills of the profile”.

Depending on the choice of an *R*-implication or an *S*-implication, the meaning of this query will vary significantly. With an *R*-implication, the levels of the skills in the profile play the role of thresholds (according to the meaning of *R*-implications mentioned in Section 3). The profile is seen as a prototype, i.e., the description of an employee with the minimal levels of skills to be completely satisfactory. Any actual candidate is penalized as soon as he/she has (at least) one skill with a level under the objective appearing in the prototype. The choice of different *R*-implications impacts the behavior of the system when the threshold is not reached. For instance, the user can make his/her choice on the basis of the ordering between some $R$-implications:

$$(p \Rightarrow_{Gö} q) \leqslant (p \Rightarrow_{Gg} q) \leqslant (p \Rightarrow_{Lu} q).$$

When an $S$-implication is used, the profile accounts for the importance of each skill, or alternatively, the extent to which a skill is considered more or less critical. As a consequence any non-completely important skill will confer a guaranteed degree of satisfaction whatever the level attained by the candidate. For the sake of convenience, but this is not strictly mandatory with the definition (formula (8)) adopted in this paper, at least one of the skills of the profile must be completely important. Here again, the choice of the $S$-implication may be done according to the ordering of some $S$-implications:

$$(p \Rightarrow_{K-D} q) \leqslant (p \Rightarrow_{Rb} q) \leqslant (p \Rightarrow_{Lu} q).$$

**Example 6.** Let us consider the database with the fuzzy relations curriculum ($c$) and profile ($p$) whose respective schemas are $C$ (candidate, skills) and $P$ (skills) with the following extensions:

| $c$ | Candidate | Skills | $\mu$ |
|---|---|---|---|
| | John | A | 1 |
| | John | B | 0.6 |
| | John | C | 0.4 |
| | Peter | A | 0.8 |
| | Peter | B | 1 |

| $p$ | Skills | $\mu$ |
|---|---|---|
| | A | 1 |
| | B | 0.8 |
| | C | 0.2 |

The division of $c$ by $p$ using Gödel (respectively Goguen) implication returns {0.6/John} (respectively {0.75/John}), whereas the use of Kleene–Dienes (respectively Reichenbach) leads to {0.6/John, 0.8/Peter} (respectively {0.68/John, 0.8/Peter}). This example illustrates the fact that a candidate which is eliminated with an $R$-implication (here Peter does not possess at all skill $C$ and he is discarded with the two considered $R$-implications), can be the best one with an $S$-implication (Peter is only somewhat penalized since competence $C$ is not considered highly important in this context).

## 7. Conclusion

The topic of this paper is the extension of the division to fuzzy relations. The key point dealt with concerns the properties of the result delivered by different approaches to the extended division. More precisely, we are interested in assessing whether the result is a quotient or not, i.e., the largest fuzzy relation which, once composed with the divisor, does not exceed the dividend. Such a property is a characteristic of the result of the division of integers and justifies the appropriateness of the term "division".

Starting with the definition of the division of regular relations which calls on an inclusion, three main lines of extension are envisaged depending on the replacement of the inclusion by a degree of inclusion based on: (i) an $R$-implication, (ii) an $S$-implication or (iii) a ratio of cardinalities. It turns out that the first two approaches constitute a sound extension, because both implications (hereafter denoted by $\Rightarrow_f$) can be expressed under a residuated form of the type:

$$p \Rightarrow_f q = \sup\{u \in [0, 1] \mid cnj(p, u) \leqslant q\}$$

where $cnj(a, b)$ is a conjunction operator generalizing the usual one. Such an expression guarantees that the result of the division is both maximal and such that its product with the divisor is included in the dividend. More precisely, when an $R$-implication is used for the extended division, the Cartesian product serving for the characterization must be performed with the triangular norm generating the $R$-implication. For $S$-implications, things are somewhat similar, except that the Cartesian product has to be done with a specific conjunction operator, called a non-commutative conjunction. In addition, this works only for fuzzy implications generated by a continuous norm (or co-norm). The approach founded on the use of a degree of inclusion expressing a ratio of cardinalities does not deliver a quotient in general, whatever the norm used to compute the ratio on the one hand and the norm or the non-commutative conjunction used for the Cartesian product on the other hand.

This work opens a number of perspectives. In particular, the division considered so far can be called a non-fuzzy one since only the operand relations are fuzzy. An orthogonal approach for extending the division would be to soften the universal quantifier so as to define a truly fuzzy division based on the fuzzy linguistic quantifier "almost all" (in the spirit of [8,13]). The question would then be to determine under which assumptions the result returned by such an approximate division is a quotient.

The same type of question would arise if the operands of the division operation are no longer relations, but multi-relations (i.e., relations containing duplicates), or even fuzzy multi-relations.

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